Further generalization of the addition formula of Frobenius-Stickelberger to higher genus Abelian functions

(joint work with John Christopher Eilbeck and Matthew England)

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The Double Gamma function and the Weierstrass $\sigma$ function


$$\sigma(z) = e^{-\mu z - \nu \frac{z^2}{2}} \cdot z \cdot \frac{\prod \Gamma_2^{-1}(z | \pm \omega_1, \pm \omega_2)}{\prod \Gamma_1^{-1}(z | \pm \omega_1) \prod \Gamma_1^{-1}(z | \pm \omega_2)},$$
Main references

- C.Hermite: *Extrait d’une lettre de M. Ch. Hermite adressée à M. L. Fuchs*. Crelle J., 82(1877)
- Y.Ônishi: *Determinant formulae in Abelian functions for a general trigonal curve of degree five*. Computational Methods and Function Theory, 11(2011)
Introduction

Let $\wp(u)$ and $\sigma(u)$ be the Weierstrass functions satisfying

\[ \wp'(u)^2 = 4\wp(u)^3 - g_2 \wp(u) - g_3, \]

\[ \sigma(u) = u \exp \left\{ \int_0^u \int_0^u \left( \wp(u) - \frac{1}{u^2} \right) dud\right\}, \quad \wp(u) = -\frac{d^2}{du^2} \log \sigma(u). \]

Then we have ((Hermite and) Frobenius-Stickelberger, 1877)

\[ \frac{\sigma(u + v) \sigma(u - v)}{\sigma(u)^2 \sigma(v)^2} = \wp(v) - \wp(u) \left( \begin{array}{c} 1 \\ \wp(u) \end{array} \right) \left( \begin{array}{c} 1 \\ \wp(v) \end{array} \right), \]

\[ \sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)}) \]

\[ \frac{1}{\prod_{j=1}^n \sigma(u^{(j)})^n} = \frac{1}{\prod j!} \left| \begin{array}{cccc} 1 & \wp(u^{(1)}) & \wp'(u^{(1)}) & \cdots & \wp(n-2)(u^{(1)}) \\ 1 & \wp(u^{(2)}) & \wp'(u^{(2)}) & \cdots & \wp(n-2)(u^{(2)}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(u^{(n)}) & \wp'(u^{(n)}) & \cdots & \wp(n-2)(u^{(n)}) \end{array} \right|. \]

These formulae correspond to the canonical involution $v \mapsto -v$.

Today I will talk on an extreme and elaborate generalization of these addition formulae.
I-1. The most general genus one curve

To step up higher genus cases smoothly, we reformulate the equalities for genus 1 case. We start with the most general genus one curve $C : f(x, y) = 0$ (not with $\wp(u)$), where

$$f(x, y) = y^2 + (\mu_1 x + \mu_3) y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6),$$

$$\text{wt}(x) = -2, \quad \text{wt}(y) = -3, \quad \text{wt}(\mu_j) = -j,$$

with the point $\infty$ at infinity. Then

$$H^1_{\text{dR}}(C/Q[\mu]) \cong \left\{ \frac{h(x, y) \, dx}{f_y(x, y)} \bigg| h(x, y) \in Q[\mu][x, y] \right\} / dQ[\mu][x, y]$$

$$= Q[\mu] \frac{dx}{f_y} + Q[\mu] \frac{xdx}{f_y} \quad (= Q[\mu] \omega + Q[\mu] \eta.)$$

(Note that $f_x(x, y) \, dx + f_y(x, y) \, dy = 0$.)

Let $x(u)$ and $y(u)$ be the inverse functions defined by

$$u = \int_{\infty}^{(x(u), y(u))} \omega.$$

Then

$$x(u) = \frac{1}{u^2} + \cdots, \quad y(u) = -\frac{1}{u^3} + \cdots.$$
I-2. Sigma function for the most general genus 1 curve

The sigma function \( \sigma(u) \) associate to the genus 1 curve is

\[
\sigma(u) = \left( \frac{2\pi}{\omega'} \right)^{1/2} \Delta^{-\frac{1}{8}} \cdot \exp \left( -\frac{1}{2} \omega'^{-1} \eta'' u^2 \right) \cdot \theta \left[ \frac{1}{2} \right] (\omega'^{-1} u, \omega''/\omega'),
\]

where \( \Delta = \text{the discriminant of } C \),

\[
\begin{bmatrix}
\omega' & \omega'' \\
\eta' & \eta''
\end{bmatrix} = \begin{bmatrix}
\int_{\alpha_1} \omega & \int_{\beta_1} \omega \\
\int_{\alpha_1} \eta & \int_{\beta_1} \eta
\end{bmatrix}
\text{ with } \omega = \frac{dx}{f_y}, \quad \eta = \frac{xdx}{f_y}
\]

and \( \{ \alpha_1, \beta_1 \} \) is a symplectic basis of \( H_1(C^{\text{an}}, \mathbb{Z}) \).

However, \( \sigma(u) \) is modular invariant. Indeed we have more tightly

\[
\sigma(u) = u + \left( \left( \frac{\mu_1}{2} \right)^2 + \mu_2 \right) \frac{u^3}{3!} + \cdots \in \mathbb{Z}[\mu, \frac{\mu_1}{2}] \langle \langle u \rangle \rangle \quad (\text{Hurwitz-integral series}).
\]

We define

\[
\varphi(u) := -\frac{d^2}{du^2} \log \sigma(u).
\]

Then, we have the solution to Jacobi’s Umkehr problem

\[
\varphi(u) = x(u), \quad \varphi'(u) = 2y(u) + \mu_1 x(u) + \mu_3.
\]
I-3. The reformulated Frobenius-Stickelberger

Then we have

\[
\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)}) \div \prod_{j} \sigma(u^{(j)})^n
\]

\[
= \frac{1}{\prod_{j} j!} \left| \begin{array}{cccc}
1 & \phi(u^{(1)}) & \phi'(u^{(1)}) & \cdots & \phi^{(n-2)}(u^{(1)}) \\
1 & \phi(u^{(2)}) & \phi'(u^{(2)}) & \cdots & \phi^{(n-2)}(u^{(2)}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \phi(u^{(n)}) & \phi'(u^{(n)}) & \cdots & \phi^{(n-2)}(u^{(n)})
\end{array} \right|
\]

\[
= \left| \begin{array}{cccc}
1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\
1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots
\end{array} \right|
\]
II-1. Sample of the main results

Suppose we have defined the multivariate $\sigma(u) = \sigma(u_5, u_2, u_1)$.

The $n$-variable case (Here $n \geq 3$ for simplicity) for

$$\mathcal{C} : y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0.$$ 

Theorem. [Ô, 2011] Let $[\zeta]$ be the natural action of $\zeta = \exp^{2\pi i \frac{3}{3}}$. Then

$$\sigma \left( u^{(1)} + \cdots + u^{(n)} \right) \prod_{i < j} \sigma_1 \left( u^{(i)} + [\zeta]u^{(j)} \right) \sigma_1 \left( u^{(i)} + [\zeta]^2 u^{(j)} \right) / \prod_j \sigma_2 \left( u^{(j)} \right)^{2n-1}$$

$$= \begin{bmatrix} 1 & x(u^{(1)}) & y(u^{(1)}) & x^2(u^{(1)}) & yx(u^{(1)}) & y^2(u^{(1)}) & x^3(u^{(1)}) & yx^2(u^{(1)}) & y^2x(u^{(1)}) & \cdots \\ 1 & x(u^{(2)}) & y(u^{(2)}) & x^2(u^{(2)}) & yx(u^{(2)}) & y^2(u^{(2)}) & x^3(u^{(2)}) & yx^2(u^{(2)}) & y^2x(u^{(2)}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x(u^{(n)}) & y(u^{(n)}) & x^2(u^{(n)}) & yx(u^{(n)}) & y^2(u^{(n)}) & x^3(u^{(n)}) & yx^2(u^{(n)}) & y^2x(u^{(n)}) & \cdots \\ 1 & x(u^{(1)}) & x^2(u^{(1)}) & \cdots & x^{n-1}(u^{(1)}) \\ 1 & x(u^{(2)}) & x^2(u^{(2)}) & \cdots & x^{n-1}(u^{(2)}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x(u^{(n)}) & x^2(u^{(n)}) & \cdots & x^{n-1}(u^{(n)}) \end{bmatrix}$$

Here $u^{(j)} = ( u^{(j)}_5, u^{(j)}_2, u^{(j)}_1 )$’s are variables on the 1st stratum.
II-2. Another result \((3, 4)\)-curve, \(g = 3\)

Suppose we have defined the multivariate \(\sigma(u) = \sigma(u_5, u_2, u_1)\).

We define \(\wp\)-functions by

\[
\wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \wp_{ijk}(u) := \frac{\partial}{\partial u_k} \wp_{ij}(u), \quad \text{etc.}
\]

Then, we have a beautiful solution (explained later) to Jacobi’s Umkehr Problem, and \(\wp_{ij}(u) \in \Gamma\left(\text{Jac}(\mathcal{C}), \mathcal{O}(2\Theta^{[g-1]})\right)\), \(\wp_{ijk}(u) \in \Gamma\left(\text{Jac}(\mathcal{C}), \mathcal{O}(3\Theta^{[g-1]})\right)\), etc.

The case of the \((3, 4)\)-curve on the largest stratum in 2 variables:

**Theorem.** [EEMÔP, 2008] For \(u, v \in \mathbb{C}^3 = \kappa^{-1}(W^3)\), we have

\[
\frac{\sigma(u + v)\sigma(u - v)}{\sigma(u)^2\sigma(v)^2} = -\wp_{55}(u) + \wp_{55}(v) - \wp_{52}(u)\wp_{21}(v) + \wp_{52}(v)\wp_{21}(u)
\]

\[
- \wp_{51}(u)\wp_{22}(v) + \wp_{51}(v)\wp_{22}(u) - \frac{1}{3} (\wp_{11}(u) Q_{5111}(v) - \wp_{11}(v) Q_{5111}(u))
\]

\[
+ \frac{1}{3} \mu_1 (\wp_{52}(u)\wp_{11}(v) - \wp_{52}(v)\wp_{11}(u)) + \mu_1 (\wp_{51}(u)\wp_{21}(v) - \wp_{51}(v)\wp_{21}(u))
\]

\[
- \frac{1}{3} (\mu_1^2 - \mu_2) (\wp_{51}(u)\wp_{11}(v) - \wp_{51}(v)\wp_{11}(u)) - \frac{1}{3} \mu_8 (\wp_{11}(u) - \wp_{11}(v)),
\]

where \(Q_{5111} = \wp_{5111} - 6\wp_{51}\wp_{11}\).
II-3. One more example

(3, 4)-curve, \( g = 3 \)

Theorem. [EEMÔP] (2008)

\[
\frac{\sigma(u + v) \sigma(u + [\zeta]v) \sigma(u + [\zeta^2]v)}{\sigma(u)^3 \sigma(v)^3} = R(u, v) + R(v, u),
\]

where

\[
R(u, v) = -\frac{1}{3} \wp_{51}(u) \frac{\partial}{\partial u_1} Q_{5111}(v) - \frac{3}{4} \wp_{21}(u) \wp_{552}(v) - \frac{1}{2} \wp_{555}(u)
\]

\[
+ \frac{1}{4} \wp_{522}(u) \wp^{[55]}(v) - \frac{1}{4} \wp_{222}(u) \wp^{[52]}(v) + \frac{1}{12} \frac{\partial}{\partial u_1} Q_{5111}(u) \wp^{[55]}(v)
\]

\[
+ \frac{1}{2} \wp_{111}(u) \wp^{[22]}(v) - \frac{1}{4} \mu_1 \wp_{111}(u) \wp^{[52]}(v)
\]

\[
+ \frac{1}{2} \mu_6 \wp_{51}(u) \wp_{111}(v) - \frac{1}{4} \mu_9 \wp_{21}(u) \wp_{111}(v) - \frac{1}{2} \mu_{52} \wp_{111}(u)
\]

with

\[
\wp^{[ij]} = \text{“the determinant of the } (i, j)\text{-}(\text{complementary}) \text{ minor of } [\wp_{ij}]_{3 \times 3} \text{”}.
\]
Meta-mathematics on the generalization

In order to generalize the classical Frobenius-Stickelberger formula there are following three “Linearly Independent Directions”:

(1) Going to higher genus case;
(2) Involving Galois conjugates, especially involving an automorphism;
(3) Changing the strata on which the formula is alive;

There are various “Linear Combinations” of them.

The theory which I will talk about today is special for functions on Jacobian varieties, but not on Abelian varieties in general.
We define the sigma function $\sigma(u)$ for $\mathcal{C} : y^2 = x^5 + \mu_4x^3 + \mu_6x^2 + \mu_8x + \mu_{10}$.

$$H^1_{dR}(\mathcal{C}/\mathbb{Q}[\mu]) \cong \mathbb{Q}[\mu] \frac{dx}{f_y} + \mathbb{Q}[\mu] \frac{x \, dx}{f_y} + \mathbb{Q}[\mu] \frac{(3x^3 + \mu_4x) \, dx}{f_y} + \mathbb{Q}[\mu] \frac{x^2 \, dx}{f_y}$$

$$= \mathbb{Q}[\mu] \omega_3 + \mathbb{Q}[\mu] \omega_1 + \mathbb{Q}[\mu] \eta_3 + \mathbb{Q}[\mu] \eta_1.$$ 

Let $\omega', \omega'', \eta'$ and $\eta''$ be the period matrices of size $2 \times 2$ with respect to the basis $\omega_3, \omega_1, \eta_3, \eta_1$ and any symplectic basis of $H_1(\mathcal{C}^\text{an}, \mathbb{Z})$.

The sigma function $\sigma(u)$ is defined by

$$\sigma(u) = \sigma(u_3, u_1) = \left(\frac{2\pi}{\omega'}\right)^{2/2} \Delta^{-\frac{1}{8}} \exp \left(-\frac{1}{2} u \omega'^{-1} \eta' u\right) \cdot \vartheta \left[\begin{array}{c} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (\omega'^{-1} u, \omega''/\omega'),$$

which is a modular invariant entire function on $\mathbb{C}^2$ and a quite natural generalization of Weierstrass sigma function.

$$\sigma(u_3, u_1) = u_3 - 2 \frac{u_1^3}{3!} - 4\mu_4 \frac{u_1^7}{7!} - 2\mu_4 \frac{u_3u_1^4}{4!} + 64\mu_6 \frac{u_1^9}{9!} - 8\mu_6 \frac{u_3u_1^6}{6!}$$

$$- 2\mu_6 \frac{u_3^2u_1^3}{2!3!} + \mu_6 \frac{u_3^3}{3!} + \cdots \in \mathbb{Z}[\mu] \langle \langle u_3, u_1 \rangle \rangle.$$
III-2. Characterization of the $\sigma(u)$ for genus 2

We define a $\mathbb{R}$-bilinear form $L(\ , \ ) : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ by

$$L(u, v) = u^t(\eta'v' + \eta''v'),$$

where $v = \omega'v' + \omega''v''$ with $v, v'' \in \mathbb{R}^2$,

which is $\mathbb{C}$-linear on the 1st space and

the map $(\ell, k) \mapsto L(\ell, k) - L(k, \ell)$ on $\Lambda \times \Lambda$ is $2\pi i \mathbb{Z}$-valued,

The function $\sigma(u) = \sigma(u_3, u_1)$ is characterized (up to a multiplicative constant) by the following properties:

(i) $\sigma(u + \ell) = \chi(\ell) \sigma(u) \exp L(u + \frac{1}{2}\ell, \ell)$, $u \in \mathbb{C}^2$, $\ell \in \Lambda$,

with $\chi(\ell) \in \{\pm 1\}$ satisfying

$$\chi(\ell + k) = \chi(\ell)\chi(k) \exp \frac{1}{2}[L(\ell, k) - L(k, \ell)];$$

(ii) The set of zeroes of $u \mapsto \sigma(u)$ is exactly the canonical image $\Theta^{[2-1]}$ of $\mathcal{C} = \text{Sym}^{2-1}\mathcal{C}$, which is of order 1.
III-3. Frobenius-Stickelberger in genus 2  (1/2)

\[ \kappa^{-1} \iota(C) \xrightarrow{\kappa} C^2 \xrightarrow{\iota} C^2/\Lambda \]

\[ \iota : (x, y) \mapsto u = \int_{\infty}^{(x(u), y(u))} (\omega_3, \omega_1) \mod \Lambda. \]

\[ \wp_{11}(u + v) = -x(u) - x(v), \quad \wp_{13}(u + v) = x(u) x(v) \]

for \( u, v \in \kappa^{-1}(\iota(C)) \) (The solu. to Jacobi’s Umkehr Problem).

**Theorem.** [Ô, 2012] Let \( \sigma_1(u) = \frac{\partial}{\partial u_1} \sigma(u_3, u_1). \)

Let \( n \geq 2 \) and \( u^{(1)}, \ldots, u^{(n)} \) be variables on \( \kappa^{-1} \iota(C) \). Then we have

\[
\sigma(u^{(1)} + u^{(2)} + \cdots + u^{(n)}) \prod_{i<j} \sigma(u^{(i)} - u^{(j)}) \bigg/ \prod_j \sigma_1(u^{(j)})^n
\]

\[
= - \begin{vmatrix}
1 & x(u^{(1)}) & x^2(u^{(1)}) & y(u^{(1)}) & yx(u^{(1)}) & x^3(u^{(1)}) & \cdots \\
1 & x(u^{(2)}) & x^2(u^{(2)}) & y(u^{(2)}) & yx(u^{(2)}) & x^3(u^{(2)}) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x(u^{(n)}) & x^2(u^{(n)}) & y(u^{(n)}) & yx(u^{(n)}) & x^3(u^{(n)}) & \cdots
\end{vmatrix}.
\]

**Proof.**

\[
0 = \sigma(v) = v_3 - \frac{1}{3} v_1^3 + \cdots,
\]

\[
\sigma(u + v) = \sigma_1(u)v_1 + \sigma_3(u)v_3 + \sigma_{11}(u)v_1^2 + \cdots.
\]
Define

\[ \wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u). \]

I realized the last formula from H.F. Baker’s formulae:

\[
- \frac{\sigma(u + v) \sigma(u - v)}{\sigma(u)^2 \sigma(v)^2} = \wp_{33}(u) - \wp_{33}(v) + \wp_{13}(u) \wp_{11}(v) - \wp_{11}(u) \wp_{13}(v).
\]

Bringing \( v \rightarrow \) a point \( \in \kappa^{-1}(\iota(\mathcal{C})) \) after multiplying \( \frac{\sigma(v)^2}{\sigma_1(v)^2} \),

\[
\frac{\sigma(u + v) \sigma(u - v)}{\sigma(u)^2 \sigma_1(v)^2} = -x(v)^2 + \wp_{13}(u) - x(v) \wp_{11}(u). \quad (\text{D. Grant})
\]

Bringing \( u \rightarrow \) a point \( \in \kappa^{-1}(\iota(\mathcal{C})) \) after multiplying \( \frac{\sigma(u)^2}{\sigma_1(u)^2} \),

\[
\frac{\sigma(u + v) \sigma(u - v)}{\sigma_1(u)^2 \sigma_1(v)^2} = x(u) - x(v).
\]

This is the initial case of the formula in the last page.
IV-1. Higher genus curves

For coprime positive integers $q > d$, let $C$ be the curve defined by

$$f(x, y) = 0$$

with

$$f(x, y) = y^d - x^q + \sum_{i,j: dq > iq+jd} (\text{some coeff.}) x^i y^j, \quad (\text{wt}(x) = -d, \; \text{wt}(y) = -q)$$

adjoining unique point $\infty$ at infinity. Call this $(d, q)$-curve.

If $C$ is non-singular, then its genus is given by

$$g = \frac{(d-1)(q-1)}{2}.$$ 

For example,

$$\begin{cases} f(x, y) = y^2 + (\mu_1 x + \mu_3) y - (x^3 + \mu_2 x^2 + \mu_4 x + \mu_6), \\
\quad \text{wt}(x) = -2, \; \text{wt}(y) = -3, \; \text{wt}(\mu_j) = -j. \\
\end{cases}$$

$$\begin{cases} f(x, y) = y^3 + (\mu_1 x + \mu_4) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) \\
\quad \text{wt}(x) = -3, \; \text{wt}(y) = -4, \; \text{wt}(\mu_j) = -j. \\
\end{cases}$$

\ldots \ldots \ldots
IV-2. Weierstrass gaps at $\infty$ of the curve $\mathcal{C}$

Let $w_1, \ldots, w_g$ be the Weierstrass gap sequence at $\infty$ of

$$\mathcal{C} : y^d + \cdots = x^q + \cdots.$$ 

For example,

$(2,3)$-curve .......... $w_1 = 1$.

$(2,2g+1)$-curve ... $(w_1, w_2, \cdots, w_g) = (1, 3, \cdots, 2g+1)$.

$(3,4)$-curve .......... $(w_1, w_2, w_3) = (1, 2, 5)$.

$(3,5)$-curve .......... $(w_1, w_2, w_3, w_4) = (1, 2, 4, 7)$.

Let us fix a vector $\vec{\omega} = (\omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1})$ consists of the “natural” basis of $\Gamma(\mathcal{C}, \Omega^1)$ with $\text{wt}(\omega_i) = w_j$.

Example. For the $(2,7)$-curve

$$f(x, y) = y^2 + (\mu_1 x^3 + \mu_3 x^2 + \mu_5) y$$

$$- (x^7 + \mu_2 x^6 + \mu_4 x^5 + \mu_6 x^4 + \mu_8 x^3 + \mu_{10} x^2 + \mu_{12} x + \mu_{14}) = 0,$$

the vector $\vec{\omega}$ consists of $\omega_5 = \frac{dx}{f_y(x,y)}$, $\omega_3 = \frac{x dx}{f_y(x,y)}$, $\omega_1 = \frac{x^2 dx}{f_y(x,y)}$. 


Example. For the \((3,4)\)-curve
\[
\begin{align*}
f(x, y) &= y^3 + (\mu_1 x + \mu_4) y^2 + (\mu_2 x^2 + \mu_5 x + \mu_8) y \\
&\quad - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0,
\end{align*}
\]
the vector \(\vec{\omega}\) consists of
\[
\begin{align*}
\omega_5 &= \frac{dx}{f_y(x,y)}, \quad \omega_2 = \frac{x dx}{f_y(x,y)}, \quad \omega_1 = \frac{y dx}{f_y(x,y)}.
\end{align*}
\]

Using \(\vec{\omega} = (\omega_{w_g}, \omega_{w_g-1}, \cdots, \omega_{w_1})\), define the period lattice \(\Lambda = \{ \int \vec{\omega} \} \subset \mathbb{C}^g\).

We define, for each integer \(k \geq 0\),
\[
\iota : \text{Sym}^k(\mathcal{C}) \to \mathbb{C}^g/\Lambda = \text{Jac}(\mathcal{C})
\]
\[
(P_1, \cdots, P_k) \mapsto \sum_{j=1}^k \int_{P_j} \vec{\omega} \mod \Lambda.
\]

We denote the mod \(\Lambda\) map by \(\kappa : \mathbb{C}^g \to \mathbb{C}^g/\Lambda\).

We denote \(W^{[k]} = \iota(\text{Sym}^k(\mathcal{C}))\). Then \(W^{[1]} \cong \mathcal{C}\). Let
\[
\Theta^{[k]} = [-1]W^{[k]} \cup W^{[k]}.
\]
IV-4. The stratification

Summing up, we have the following stratification:

\[ \Lambda \subset \kappa^{-1}(\Theta^{[1]}) \subset \kappa^{-1}(\Theta^{[2]}) \subset \cdots \subset \kappa^{-1}(\Theta^{[g-1]}) \subset \kappa^{-1}(\Theta^{[g]}) = C^g. \]

\[ \downarrow \kappa \quad \downarrow \kappa \quad \downarrow \kappa \quad \downarrow \kappa \]

\[ 0 \in \Theta^{[1]} \quad \subset \Theta^{[2]} \quad \subset \cdots \subset \Theta^{[g-1]} \quad \subset \Theta^{[g]} = C^g/\Lambda \]

\[ \uparrow \iota \quad \uparrow \iota \quad \uparrow \iota \quad \uparrow \iota \]

\[ \infty \in C = \text{Sym}^1 C \subset \text{Sym}^2 C \subset \cdots \subset \text{Sym}^{g-1} C \subset \text{Sym}^g C \]

We note that Jacobi’s theorem implies

\[ \Theta^{[g-1]} = W^{[g-1]}. \]

We shall define afterward an important function \( \sigma_{[k]}(u) \) (a higher derivative of \( \sigma(u) \)), which is useful on the \( k \)-th stratum \( \kappa^{-1}(\Theta^{[k]}). \)
VI-5. de Rham cohomology and its symplectic structure

On the 1st de Rham cohomology

\[
H^1_{\text{dR}}(\mathcal{C}/\mathbb{Q}[\mu]) = \left\{ \frac{h(x,y) \, dx}{f_y(x,y)} \middle| h(x,y) \in \mathbb{Q}[\mu][x,y] \right\} \bigg/ d\mathbb{Q}[\mu][x,y],
\]

\[
\left( \supset \Gamma(\mathcal{C}, \Omega^1) \right)
\]

we have the following symplectic product \( \star \):

For \( \omega, \eta \in H^1_{\text{dR}}(\mathcal{C}/\mathbb{Q}[\mu]) \),

\[
\omega \star \eta = \sum_{P} \text{Res}_P \left( \int_{P}^{\infty} \omega \right) \eta(P) = \text{Res}_{P=\infty} \left( \int_{\infty}^{P} \omega \right) \eta(P).
\]

There is a "concise" symplectic basis of \( H^1_{\text{dR}}(\mathcal{C}/\mathbb{Q}[\mu]) \):

\[
\omega_{w_g}, \omega_{w_{g-1}}, \ldots, \omega_{w_1}, \eta_{w_g}, \eta_{w_{g-1}}, \ldots, \eta_{w_1},
\]

where \( w_j \) stands for the weight (or the negative of weight).
IV-6. The sigma function for a higher genus curve

The sigma function \( \sigma(u) \) for \( C \) is defined by using the symplectic basis 
\[ \{ \omega_{w_g}, \omega_{w_{g-1}}, \cdots, \omega_{w_1} \} \cup \{ \eta_{w_g}, \eta_{w_{g-1}}, \cdots, \eta_{w_1} \} \] of \( H^1_{\text{dR}}(C/\mathbb{Q}[\mu]) \) and any symplectic basis of 
\( H_1(C^{\text{an}}, \mathbb{Z}) \). It is an entire function on \( C^g \) with \( g \) variables \( u = (u_{w_g}, \cdots, u_{w_1}) \), and it is a quite natural generalization of the Weierstrass sigma function.

Example. If \( C \) is \((3,4)\)-curve, then
\[
\sigma(u) = \sigma(u_5, u_2, u_1) = (u_5 - u_1u_2^2 + \frac{1}{20}u_1^5) + \left( \frac{1}{12}u_1u_2^4u_2 - \frac{1}{3}u_1u_2^3 \right) + \cdots.
\]

We define \( \mathbb{R} \)-bilinear form \( L(\ , \ ) : C^2 \times C^2 \to \mathbb{C} \) by
\[
L(u, v) = u^t(\eta'v' + \eta''v'), \quad \text{where } v = \omega'v' + \omega''v'' \quad \text{with } v, \ v'' \in \mathbb{R}^g,
\]
which is \( \mathbb{C} \)-linear on the 1st space and the map \((\ell, k) \mapsto L(\ell, k) - L(k, \ell)\) on \( \Lambda \times \Lambda \) is \( 2\pi i \mathbb{Z} \)-valued,

The function \( \sigma(u) = \sigma(u_{w_g}, \cdots, u_{w_1}) \) is characterized (up to non-zero multiplicative constant) by

(i) \( \sigma(u + \ell) = \chi(\ell)\sigma(u) \exp L(u + \frac{1}{2}\ell, \ell) \) for \( u \in C^g, \ell \in \Lambda \),
with \( \chi(\ell) \in \{ \pm 1 \} \) satisfying \( \chi(\ell + k) = \chi(\ell)\chi(k) \exp \frac{1}{2}[L(\ell, k) - L(k, \ell)] \);

(ii) The set of zeroes of \( u \mapsto \sigma(u) \) is exactly on \( \Theta^g \cup [-1]\Theta^g \),
which is of order 1. Here \( \Theta^g \) is the canonical image of \( \text{Sym}^{g-1}C \).
V-1. On the largest stratum \((3, 4)\)-curve, \(g = 3\)

We define \(\wp\)-functions by

\[ \wp_{ij}(u) := -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u), \quad \wp_{ijk}(u) := \frac{\partial}{\partial u_k} \wp_{ij}(u), \quad \text{etc.} \]

Then \(\wp_{ij}(u) \in \Gamma(Jac(C), \sigma(2\Theta^{[g-1]})\), \(\wp_{ijk}(u) \in \Gamma(Jac(C), \sigma(3\Theta^{[g-1]})\), \ etc.

The case of the \((3, 4)\)-curve on the largest stratum in 2 variables:

Theorem. [EEMÔP] (2008)

For \(u, v \in C^3 = \kappa^{-1}(W^3)\), we have

\[
\frac{\sigma(u + v) \sigma(u - v)}{\sigma(u)^2 \sigma(v)^2} = -\wp_{55}(u) + \wp_{55}(v) - \wp_{52}(u) \wp_{21}(v) + \wp_{52}(v) \wp_{21}(u) \\
- \wp_{51}(u) \wp_{22}(v) + \wp_{51}(v) \wp_{22}(u) - \frac{1}{3} (\wp_{11}(u) Q_{5111}(v) - \wp_{11}(v) Q_{5111}(u)) \\
+ \frac{1}{3} \mu_1 (\wp_{52}(u) \wp_{11}(v) - \wp_{52}(v) \wp_{11}(u)) + \mu_1 (\wp_{51}(u) \wp_{21}(v) - \wp_{51}(v) \wp_{21}(u)) \\
- \frac{1}{3} (\mu_1^2 - \mu_2) (\wp_{51}(u) \wp_{11}(v) - \wp_{51}(v) \wp_{11}(u)) - \frac{1}{3} \mu_8 (\wp_{11}(u) - \wp_{11}(v)),
\]

where \(Q_{5111} = \wp_{5111} - 6 \wp_{51} \wp_{11} \).
V-2. On the largest stratum for the purely trigonal curve

Theorem. [EEMÔP] (2008)

For $\mathcal{C}: f(x, y) = y^3 - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0$ with the canonical automorphism $[\zeta]: (x, y) \mapsto (x, \zeta y)$ of $\zeta = \exp(2\pi i/3)$, we have

$$\frac{\sigma(u + v) \sigma(u + [\zeta]v) \sigma(u + [\zeta]^2v)}{\sigma(u)^3 \sigma(v)^3} = R(u, v) + R(v, u),$$

where

$$R(u, v) = -\frac{1}{3} \varphi_{51}(u) \frac{\partial}{\partial u_1} Q_{5111}(v) - \frac{3}{4} \varphi_{21}(u) \varphi_{552}(v) - \frac{1}{2} \varphi_{555}(u)$$
$$+ \frac{1}{4} \varphi_{522}(u) \varphi^{[55]}(v) - \frac{1}{4} \varphi_{222}(u) \varphi^{[52]}(v) + \frac{1}{12} \frac{\partial}{\partial u_1} Q_{5111}(u) \varphi^{[55]}(v)$$
$$+ \frac{1}{2} \varphi_{111}(u) \varphi^{[22]}(v) - \frac{1}{4} \mu_1 \varphi_{111}(u) \varphi^{[52]}(v)$$
$$+ \frac{1}{2} \mu_6 \varphi_{51}(u) \varphi_{111}(v) - \frac{1}{4} \mu_9 \varphi_{21}(u) \varphi_{111}(v) - \frac{1}{2} \mu_{52} \varphi_{111}(u)$$

with

$$\varphi^{[ij]} = \text{“the determinant of the \((i, j)\)-(complementary) minor of \(\varphi_{ij}\)\(3 \times 3\)”.}$$
VI-1. Higher derivatives of the sigma function

We define, for the multi-index $I = \mathbb{I}^n$ with respect to $\{w_g, \cdots, w_1\}$ defined in the next page, or for arbitrary multi-index $I$,

$$\sigma_I(u) = \left( \prod_{j \in I} \frac{\partial}{\partial u_j} \right) \sigma(u).$$

Examples. If $(d, q) = (3, 4)$ then $b = \mathbb{I}^2 = \{1\}$ and $\sharp = \mathbb{I}^1 = \{2\}$, and

$$\sigma_b(u) = \sigma_1(u) = \frac{\partial}{\partial u_1} \sigma(u_5, u_2, u_1),$$

$$\sigma_{\sharp}(u) = \sigma_2(u) = \frac{\partial}{\partial u_2} \sigma(u_5, u_2, u_1).$$

We define $\sigma_{\mathbb{I}^0}(u) = 1$, a constant function.
### Table of $\mathbb{H}^n$

<table>
<thead>
<tr>
<th>$(d, p)$</th>
<th>$g$</th>
<th>$\mathbb{H} = \mathbb{H}^1$</th>
<th>$b = \mathbb{H}^2$</th>
<th>$\mathbb{H}^3$</th>
<th>$\mathbb{H}^4$</th>
<th>$\mathbb{H}^5$</th>
<th>$\mathbb{H}^6$</th>
<th>$\mathbb{H}^7$</th>
<th>$\mathbb{H}^8$</th>
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<td>${ 3, 7 }$</td>
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<td>${ 1, 5, 9 }$</td>
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VI-3. Table of $\mathbb{H}^n$

We explain by an example: $(d, q) = (3, 7), \ g = 6$. Write a $g \times g = 6 \times 6$ table as follows. We first write the Weierstrass gap sequence with respect to $(d, q)$ on the last column, namely,

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</tbody>
</table>
VI-4. Table of $\mathbb{H}^n$

Then, put into other boxes naturally increasing non-negative integers as follows:

$$
\begin{array}{cccccc}
6 & 7 & 8 & 9 & 10 & 11 \\
3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 2 & 3 & 4 & \\
0 & 1 & 2 & \\
0 & 1 & \\
\end{array}
$$
VI-5. Table of $\mathfrak{H}^n$

If we wish to get $\mathfrak{H}^n = \mathfrak{H}^2$, extract $(g - n) \times (g - n) = 4 \times 4$ minor on the lower right corner. and Remove all rows and columns including 0.

\[
\begin{array}{cccccc}
6 & 7 & 8 & 9 & 10 & 11 \\
3 & 4 & 5 & 6 & 7 & 8 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 2 & 3 & 4 & \\
0 & 1 & 2 & & & \\
0 & 1 & & & & \\
\end{array}
\rightarrow
\begin{array}{cccc}
2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 \\
0 & 1 & 2 & \\
0 & 1 & & & \\
\end{array}
\rightarrow
\begin{array}{cc}
2 & 5 \\
1 & 4 \\
\end{array}
\]

Finally, by reading the numbers on the off-diagonal, we have

\[
\mathfrak{H}^2 = \{1, 5\} \quad \text{and} \quad \sigma_{\mathfrak{H}^2}(u) = \sigma_{1,5}(u) = \frac{\partial^2}{\partial u_1 \partial u_5} \sigma(u).
\]
VI-3. Properties of the satellite sigma functions (The most important page!)

The set of higher derivatives of the $\sigma(u)$

\[
\{ \kappa^{-1}(\Theta^{[n]}) \ni u \mapsto \sigma_{n}(u) \mid 0 \leq n \leq g - 1 \}
\]

the satellite sigma functions for $C$. They have the following very nice properties:

(i) $\sigma_{n}(u + \ell) = \chi(\ell) \sigma_{n}(u) L(u + \frac{1}{2}\ell, \ell), \ u \in \kappa^{-1}(\Theta^{[n]}), \ \ell \in \Lambda$.

(ii) If $u \in \kappa^{-1}(W^{[n]} \setminus W^{[n-1]})$, then the function $\kappa^{-1}(W^{[1]}) \ni v \mapsto \sigma_{n+1}(u + v)$ has a zero at $\Lambda$ of order $w_{g-n} - g + n + 1$,
and other $g - (w_{g-n} - g + n + 1)$ zeroes elsewhere mod $\Lambda$.
Moreover, $\sigma_{n+1}(u + v) = \pm \sigma_{n}(u) \nu_{1}^{w_{g-n} - g + n + 1} + \text{"higher terms in } v_{1}\"$.

The exact places of all zeroes of $v \mapsto \sigma_{b}(u + v) := \sigma_{2}(u + v)$ are known.

$\sigma_{n}(u) := \sigma_{1}(u) = \pm \nu_{1}^{g} + \cdots$ and this has only zero at $\Lambda$.

(iii) The set of zeroes of the function $\kappa^{-1}(W^{[n+1]}) \ni u \mapsto \sigma_{n+1}(u)$ is $\kappa^{-1}(\Theta^{[n]})$, which is of order 1.

(iv) For an index $I$, if $\text{wt}(I) < \text{wt}(\mathfrak{h}^{n})$, then $\sigma_{I}(u) = 0$ on $\kappa^{-1}(\Theta^{[n]})$.

(v) If $\text{wt}(I) = \text{wt}(\mathfrak{h}^{n})$, then the function $\sigma_{I}(u) = \text{"an integer"} \cdot \sigma_{n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$.

Proof: By certain expression of $\sigma(u)$ as the determinant of a matrix of size $\mathbf{N} \times \mathbf{N}$
(or by precise observation of power series expansions).
VII-1. Guide Function

We may extend this class of addition formulae by considering more general map

$$\varphi : \mathbb{C} \longrightarrow \mathbb{P}^1$$

which belongs to $\mathbb{Z}[\mu_1, \mu_2, \cdots, \mu_6][x(u), y(u)]$, and of homogeneous weight. We suppose the coefficient of the lowest weight term w. r. t. $x(u)$ and $y(u)$ is 1. Let $m \geq 2$ be the order of unique pole of $\varphi$, and $u$ be the analytic variable of $\varphi$ regarding $\mathbb{C}$ as a complex torus. Then there exist

$$u, u^*, u^{*2}, u^{*3}, \cdots, u^{*m-1} \in \mathbb{C}$$

such that these $m$ variables are generically different, vary continuously, and satisfy

$$\varphi(u) = \varphi(u^*) = \cdots = \varphi(u^{*m-1}).$$

Moreover, we may choose them as

$$u + u^* + \cdots + u^{*m-1} = 0.$$ 

Indeed $d(u + u^* + \cdots + u^{*m-1})$ can be regarded as a holomorphic 1-form on $\mathbb{P}^1$. 
Example. ([Eilbeck-England-Ô, 2014]) We take the \((2,3)\)-curve and \(y(u)\) as a guide function. \((y(u) = y(u^*) = y(u^{**}))\) Let \(u = u^{(1)}\) and \(v = u^{(2)}\) (two variable case).

Then we have the addition formula

\[
\frac{\sigma(u + v) \sigma(u + v^*) \sigma(u + v^{**})}{\sigma(u)^3 \sigma(v) \sigma(v^*) \sigma(v^{**})} = y(v) - y(-u) = y(u) + y(v) + \mu_1 x(u) + \mu_3
\]

\[
= \frac{f(x(u), Y) - f(x(u), W)}{Y - W} \bigg|_{Y = y(u), W = y(v)}.
\]

Proof. Use the following: As a function of \(u\),

\[
y(v) - y(-u) = 0 \iff u = -v, -v^*, \text{ or } -v^{**};
\]

\[
y(v) - y(-u) = \infty \iff u = 0;
\]

\[
u + u^* + u^{**} = 0;
\]

\[
\sigma(u) = 0 \iff u \in \Lambda.
\]

Remark. The RHS is defined over \(\mathbb{Z}[\mu] = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]\).

Remark. There is [Eilbeck-S.Matsutani-Ô, 2011] for \(y^2 + \mu_3 y = x^3 + \mu_6\).
We take the \((2, 3)\)-curve and \(x^2(u)\) as a guide function. Let \(u = u^{(1)}\) and \(v = u^{(2)}\) (two variable case). Then

**Example.** We have the addition formula

\[
\frac{\sigma(u + v) \sigma(u + v^*) \sigma(u + v^{**}) \sigma(u + v^{***})}{\sigma(u)^4 \sigma(v) \sigma(v^*) \sigma(v^{**}) \sigma(v^{***})} = x^2(u) - x^2(v).
\]

**Remark.** The RHS is defined over \(\mathbb{Z}[\mu] = \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]\).
VII-4. On the first stratum in two variables $(3,4)$-curve, $g = 3$

We define the functions $\kappa^{-1}(W[1]) \ni u \mapsto x(u), \kappa^{-1}(W[1]) \ni u \mapsto y(u)$ by

$$u = (u_{w_1}, \ldots, u_{w_1}) = \int_{\omega}^{(x(u),y(u))} \omega.$$ 

Let us take $x(u)$ be the guide function.

For a variable $v \in \kappa^{-1}(W[1])$, let $\{v, v', v''\}$ be a complete representative modulo $\Lambda$ of the inverse image of the map $v \mapsto x(v)$ such that $v'$ and $v''$ vary continuously with respect to $v$ and $v' = v'' = 0$ when $v = 0$.

Of course, $y(v), y(v'), y(v'')$ are the three roots of $f(x(v), Y) = 0$.

Lemma. [Ô] (2011) Then, for $u, v \in \kappa^{-1}(W[1])$, we have

$$\frac{\sigma_b(u + v) \sigma_b(u + v') \sigma_b(u + v'')} {\sigma^b(u)^3 \sigma^b(v) \sigma^b(v') \sigma^b(v'')} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}^2.$$ 

Here we recall that

$$\sigma_b(u) = \sigma^b_2(u) = \sigma_2(u) = \frac{\partial}{\partial u_2} \sigma(u), \quad \sigma^b_1(u) = \sigma^b_{11}(u) = \sigma_1(u) = \frac{\partial}{\partial u_1} \sigma(u).$$
Theorem. [Ô] (2011) In $n$-variable case (Here $n \geq 3$ for simplicity):

$$\sigma(u(1) + \cdots + u(n)) \prod_{i<j} \sigma_1(u(i) + u(j)) \sigma_1(u(i) + u(j)') \sigma_1(u(i) + u(j)'')$$

$$\prod_j \sigma_2(u(j))^{2n-2j+1} \sigma_2(u(j)')^{j-1} \sigma_2(u(j)'')^{j-1}$$

$$= \begin{vmatrix}
1 & x(u(1)) & y(u(1)) & x^2(u(1)) & yx(u(1)) & y^2(u(1)) & x^3(u(1)) & yx^2(u(1)) & y^2x(u(1)) & \cdots \\
1 & x(u(2)) & y(u(2)) & x^2(u(2)) & yx(u(2)) & y^2(u(2)) & x^3(u(2)) & yx^2(u(2)) & y^2x(u(2)) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x(u(n)) & y(u(n)) & x^2(u(n)) & yx(u(n)) & y^2(u(n)) & x^3(u(n)) & yx^2(u(n)) & y^2x(u(n)) & \cdots \\
1 & x(u(1)) & x^2(u(1)) & \cdots & x^{n-1}(u(1)) \\
1 & x(u(2)) & x^2(u(2)) & \cdots & x^{n-1}(u(2)) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x(u(n)) & x^2(u(n)) & \cdots & x^{n-1}(u(n))
\end{vmatrix} \cdot$$
Theorem. [EEÔ] (2014)
On the 1st stratum in 2-variables \( u \) and \( v \) with guide function \( y \) (order 4), we have

\[
\begin{align*}
\sigma_1(u + v)\sigma_1(u + v^*)\sigma_1(u + v^{**})\sigma_1(u + v^{***}) \\
\sigma_2(u)^4\sigma_2(v)\sigma_2(v^*)\sigma_2(v^{**})\sigma_2(v^{***}) \\
y(u)^2 + y(u)y(v) + y(v)^2 + (\mu_1x(u)+\mu_4)(y(u)+y(v)) + \mu_2x(u)^2 + \mu_5x(u)+\mu_8 \\
f(x(u), Y) - f(x(u), W) \bigg|_{Y=y(u), W=y(v)} = (y(v) - y(u'))(y(v) - y(u'')).
\end{align*}
\]

Remark. Of course, \( y(u) = y(u^*) = y(u^{**}) = y(u^{***}) \), \( y(u') = y(u'^*) = y(u'^{**}) = y(u'^{***}) \), \( y(u'') = y(u''^*) = y(u''^{**}) = y(u''^{***}) \).

Keys of the proof. For a fixed \( u \in \kappa^{-1}(\Theta^{[1]}) \), the map \( v \mapsto \sigma_b(u + v) \) has a zero at \( v = 0, u', u'' \) modulo \( \Lambda \) of order 1, and the map \( u \mapsto \sigma_\#(u) \) has only zero at \( u = 0 \) modulo \( \Lambda \) of order \( (g =)3 \), and no zeroes elsewhere.
Connection with multiple Gamma functions?

Recall the famous infinite product expression for the Weierstrass sigma:

\[ \sigma(u) = u \prod_{\substack{\ell \in \Lambda \\ \ell \neq 0}} \left(1 - \frac{u}{\ell}\right) \exp \left(\frac{u}{\ell} + \frac{u^2}{2\ell^2}\right). \]

This implies the connection with the double Gamma functions:

\[ \sigma(z) = e^{-\mu z - \nu z^2} \cdot z \cdot \frac{\prod \Gamma_{2}^{-1}(z \mid \pm \omega_1, \pm \omega_2)}{\prod \Gamma_{1}^{-1}(z \mid \pm \omega_1) \prod \Gamma_{1}^{-1}(z \mid \pm \omega_2)}, \]

In the higher genus case,

\[ \sigma_{\#}(u) = \sigma_{\#}(u_{w_g}, \ldots, u_{w_1}) \text{ on } \kappa^{-1} \nu(C) \text{ has zeroes of order } g, \text{ and no zeroes elsewhere.} \]

Does it have some infinite product expression?

The speaker has a dream on existence of

(1) an infinite product expression of \( \sigma_{\#}(u) \) and

(2) an infinite product expression of multivariate multiple \( \Gamma \) functions, and their connection.
VIII. Summary and Some Questions

For each curve $C$ and for each the following setting, we have an addition formula of F-S type:

1. $k \cdots$ the stratum: on the 1st stratum $\rightarrow$ by using $x(u)$ and $y(u)$;
   on the largest stratum $\rightarrow$ by using $\varphi$-functions,

2. $n \cdots$ the number of variables,

3. $\varphi \cdots$ the guide function.

Some Questions:

Q1 Is there further natural generalization?

Q2 Why the coefficients of RHS belong to $\mathbb{Z}[\mu]$? (It is obvious they belong to $\mathbb{Q}[\mu]$.)
   (If the order of the guide function is small Q2 is OK because the RHS is a determinant, etc.)

Q3 How do these formulae link with other existing mathematical world? Or some applications?

Q4 Can the general RHS be regarded as a sort of higher generalization of
   the concept of “determinant”?

Additional reference.

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Please check

http://www2.meijo-u.ac.jp/~yonishi/

Thank you very much for your attention!